ISRAEL JOURNAL OF MATHEMATICS 81 (1993), 1–30

LINEAR O-MINIMAL STRUCTURES

BY

JAMES LOVEYS AND YA'ACOV PETERZIL*

Department of Mathematics
McGill University, Montreal, QC, Canada

ABSTRACT

A linearly ordered structure $\mathcal{M} = \langle M, <, \cdots \rangle$ is called **o-minimal** if every definable subset of M is a finite union of points and intervals. Such an $\mathcal{M}$ is a CF **structure** if, roughly said, every definable family of curves is locally a one-parameter family. We prove that if $\mathcal{M}$ is a CF structure which expands an (interval in an) ordered group, then it is elementary equivalent to a reduct of an (interval in an) ordered vector space. Along the way we prove several quantifier-elimination results for expansions and reducts of ordered vector spaces.

1. Introduction

General o-minimal theories were first studied by Pillay and Steinhorn in [PS]. The classical examples include the real field, the natural numbers with the usual order, and ordered vector spaces over the rationals, reals, or any ordered division ring. It is examples of the latter sort (and their reducts) which concern us here. We remind the reader that a first-order theory T with distinguished total order $<$ is **o-minimal** if for any $M \models T$, any definable subset of M is a union of finitely many points and open intervals. We will always assume that our structures are densely ordered by $<$.

* The research for this article was begun when the authors were at Berkeley during the logic year at the Mathematical Science Research Institute. It was completed at McGill University. The research was supported by grants from NSERC and FCAR. JL would, as always, like to thank Alistair.
Received February 27, 1992 and in revised form November 2, 1992

Although o-minimal theories cannot be stable, they are close relatives of those stable theories called **strongly minimal** or those called **weakly minimal**. Our experience leads us to believe that they are closer to the latter. The models of such theories carry an exchange geometry naturally induced by the algebraic closure operation (on subsets); this is also the case for o-minimal theories. In recent stability theory, one of the crucial assumptions on the geometries so arising is that they are **locally modular**. This condition is known to separate those that are geometrically complex from those that are relatively simple. For weakly minimal theories, the assumption that local modularity occurs locally (i.e., when one restricts the geometry to the set of realizations of some strong type) is equivalent to the notion of **1-basedness**. A great deal is known about 1-based weakly minimal theories. If such theory T is nontrivial, it interprets an abelian group A, which can be described as a reduct of a weakly minimal module (see [HL]). Our results can be considered as an o-minimal version of these; we will not otherwise deal with them.

We believe that the notion corresponding to 1-basedness in the o-minimal context is what we define below as the **CF property**. Roughly, it says that if we have any uniformly definable family of partial functions, then these functions can be defined using a single parameter. Evidence for the naturality of this condition is in the results below, together with the following two results:

THEOREM 1.1 ([P1]): *Let $\mathcal{M} = \langle M; <, \cdots \rangle$ be an o-minimal structure with the CF property. Suppose that $\mathcal{M}$ has nontrivial geometry (equivalently, $\mathcal{M}$ is nondyadic). Then there is an interval of M on which a structure of a group-interval is $\mathcal{M}$-definable.*

This result is our starting point here. The notion of group-interval will be defined properly below; for the moment, think of a closed and bounded interval in an ordered group with the group operation restricted to that interval.

On the other side, there is:

THEOREM 1.2 ([P2]): *Let $\mathcal{M} = \langle M; <, \ldots \rangle$ be a reduct of the real field which does not have the CF property. Then there is an interval of M on which a real closed field is $\mathcal{M}$-definable.*

Our work will be the characterization of groups and goup-intervals satisfying the CF condition. Specifically, we will prove:

THEOREM 1.3: *Let T be an o-minimal theory satisfying the CF property.*

1. *If T is the theory of a group, then it is a reduct of $Th(\langle V;+,<,d,a\colon d\in D, a\in V_0\rangle)$, where $\langle V;+,<,d\colon d\in D\rangle$ is an ordered vector space over the ordered division ring D, and V_0 is a collection of algebraic points.*
2. *If T is the theory of a goup-interval, then T is $Th(\mathcal{M})$, for some $\mathcal{M}$ as follows. Let $\mathcal{V}$ be an ordered vector space, as in the last clause, I some interval in V and consider the structure induced by $\mathcal{V}$ on I. $\mathcal{M}$ is a reduct of this structure.*

There is actually more information contained below, including a quantifier elimination result for the reducts of ordered vector spaces. We also show that if T above is modular then we can omit the word 'reduct' in both statements. Note that this result, taken together with Theorem 1.1, gives complete information, at least locally, about nontrivial CF theories.

2. Preliminaries

2.1 O-MINIMAL STRUCTURES. We first introduce some logical conventions: The symbol $\mathcal{M}$ will be reserved for structures, and M for the universe of $\mathcal{M}$. Unless otherwise stated, we take 'definable' to mean 'definable with parameters'. In some cases we will make precise the fact that we use parameters from A by saying 'A-definable'. For $B\subseteq M$ and $a\in M$, we say that a **is in the definable closure of** B, $a\in \mathrm{dcl}(B)$, if there is a formula with parameters from B such that a is the unique element satisfying this formula. We say that a **is in the algebraic closure of** B, or $a\in \mathrm{acl}(B)$, if a is one of finitely many solutions to such a formula. We sometimes call a **an algebraic point** if it is in $\mathrm{acl}(\emptyset)$. Notice that in ordered structures the definable closure is the same as the algebraic closure. We say that $\mathcal{M}$ has **trivial geometry** if for every $A\subseteq M$ we have $\mathrm{acl}(A)=\bigcup_{a\in A}\mathrm{acl}(\{a\})$.

The first order languages we consider will always assume to contain the binary relation $<$, interpreted in all structures as a dense linear ordering. Given $\mathcal{M}$, we always refer to the topology induced by $<$ on the different M^k's. A structure $\mathcal{M}$ is called **o-minimal** if every definable subset of M is a finite union of points from M and open intervals, all of whose endpoints lie in $M\cup\{\pm\infty\}$. As was shown in [KPS], if $\mathcal{M}$ is o-minimal and $\mathcal{N}\equiv\mathcal{M}$ then $\mathcal{N}$ is also o-minimal. We refer the reader to this last paper and to [PS] for the following basic results and definitions.

The cell-decomposition theorem says that every A-definable set in an o-minimal structure is a finite union of A-definable cells (we omit here the definition of a cell). The following corollaries immediately follow:

COROLLARY 2.1: *Let $U \subseteq M^n$ be an A-definable open set and f an A-definable function from U into $\mathcal{M}$. Then U can be partitioned into finitely many A-definable sets, $U_1, \ldots, U_k$, such that f is continuous on each U_i.*

The following corollary can actually be strengthened to give quantifier elimination down to the relation $<$ and the 0-definable partial continuous functions.

COROLLARY 2.2: *Let $\mathcal{M}$ be o-minimal and $\{f_i\}_{i\in I}$ the collection of all 0-definable continuous (partial) functions from M^n into M, for various n. Then every 0-definable set is 0-definable in the language containing $<$ and the graphs of all the f_i's.*

One of the main tools in the proof of the cell-decomposition theorem is the fact that, in o-minimal structures, every definable unary function f is piecewise monotone. Namely, its domain can be partitioned into finitely many intervals and points such that on each interval f is either strictly monotone or constant. As a corollary, we have the exchange principle for the definable closure in o-minimal structures.

We define two notions of dimension for an o-minimal structure $\mathcal{M}$. As we will point out below the two are related to one another. The algebraic dimension of a subset $A \subseteq M$ over another subset $B \subseteq M$ is defined to be:

$$\dim A/B = \min\{|A'| : A' \subseteq A \text{ and for all } a \in A, a \in \mathrm{dcl}(A' \cup B)\}.$$

The topological dimension of **a definable** subset of M^k is defined as follows: Since every cell C is definably homeomorphic to an open subset of some M^n, for a unique n, we can define the dimension of C to be this n. If $U \subseteq M^k$ is a definable set then

$$\dim(U) = \text{The largest } n \text{ such that } U \text{ contains a cell of dimension } n.$$

These notions have all the properties we expect them to have. Namely, $\dim(A/B)$ is the same as the cardinality of any maximal independent (over B)

subset of A. If $U = C_1 \cup \cdots \cup C_n$ is a union of cells then $\dim(U) = \max\{\dim(C_i)\}$. Furthermore, as was pointed out in [Pi], if $\{U_{\bar{a}}\}_{\bar{a} \in M^k}$ is an A-definable family of uniformly definable sets then, for every $l \in \mathbb{N}$, the set $\{\bar{a} : \dim(U_{\bar{a}}) = l\}$ is A-definable. We also make use of the fact (see [vdD]) that for a definable set $X \subseteq M^n$, the boundary of X in M^n has dimension smaller than that of X.

The following dimension formula for tuples was established in [Pi]: If $\bar{u}, \bar{v}$ are tuples from M and $A \subseteq M$, then

$$\dim(\bar{u}\bar{v}/A) = \dim(\bar{u}/\bar{v}A) + \dim(\bar{v}/A).$$

The next definition brings together the two notions of dimensions.

Definition 2.3: Let $\mathcal{M}$ be an o-minimal structure, and assume that U is an A-definable subset of M^k for some k. We say that $\bar{u}_0$ **is generic in U over A** if $\dim(\bar{u}_0/A) = \dim(U)$.

As was noted in [Pi], if $\mathcal{M}$ is an ω-saturated structure, $A \subseteq M$ is finite and U an A-definable set, then

$$\bar{u}_0 \text{ is generic in } U \text{ over } A \text{ iff } \dim(\bar{u}_0/A) = \max\{\dim(\bar{u}/A) : \bar{u} \in U\}.$$

For U a definable subset of M^k and $\bar{u} \in U$, we say that **$V \subseteq U$ is a neighborhood of $\bar{u}$** if V contains an open set around $\bar{u}$ in the relative topology of U. It is easily seen that if $V \subseteq U$ is a neighborhood of a generic point in U, then $\dim(V) = \dim(U)$. Now if $\bar{u}_0$ is generic in U over A and if $V \subseteq U$ is an A-definable set containing $\bar{u}_0$, then the boundary of $U \setminus V$ has smaller dimension than that of U and hence $\bar{u}_0$ is an interior point of V in U. It follows that whenever $\bar{u}_0$ satisfies a first order property which can be stated using parameters from A then there is a neighborhood $V \subseteq U$ of $\bar{u}_0$ where this property holds.

Some of the work in this paper will be done under the following assumptions:

Definition 2.4: (i) A structure $\mathcal{M}$ called **modular** if for every two subsets $A, B \subseteq M$ we have

$$\dim(A/\emptyset) + \dim(B/\emptyset) = \dim(\mathrm{dcl}(A) \cap \mathrm{dcl}(B)/\emptyset) + \dim(A \cup B/\emptyset).$$

$\mathcal{M}$ is called **locally modular** if there is a tuple $\bar{a}$ from M such that $\langle \mathcal{M}, \bar{a} \rangle$ is modular.

(ii) A theory T is called **modular (locally modular)** if there is a $|T|^+$-saturated model $\mathcal{M} \models T$ which is modular (locally modular).

2.2 On definable equivalence relations. We first introduce some notation. Let U be a definable subset of M^n. For E a definable equivalence relation on U denote by $[\bar{u}]_E$ the E-equivalence class of $\bar{u}$ in U. We define the dimension of the quotient $\frac{U}{E}$ as follows:

For $k \in \{0, \ldots, \dim(U)\}$, we let $U_k^E = \{\bar{u} \in U : \dim([\bar{u}]_E) = k\}$. Notice that U_k^E is definable over the same parameters used to define U and E. We then define:

$$\dim\left(\frac{U}{E}\right) = \max\{\dim(U_k^E) - k : k \in \{0, \ldots, \dim(U)\}\}.$$

In [P1] several basic results were proved for this notion, demonstrating why this definition is natural. We will use time and again the fact that the dimension is a definable notion, in the sense that if $\{U_{\bar{a}}, E_{\bar{a}}\}_{\bar{a} \in W}$ is a definable family of sets equipped with a definable equivalence relation then for every $l \in \mathbb{N}$ $\{\bar{a} \in W : \dim\left(\frac{U_{\bar{a}}}{E_{\bar{a}}}\right) = l\}$ is a definable set. We also use the fact that $\dim(U/E) \geq 1$ iff there are infinitely many E-classes in U. For U and E as above, if $V \subseteq U$ is a definable set we will use E to denote also the restriction of the equivalence relation to V. Hence $\frac{V}{E}$ is just $\{[\bar{v}]_E \cap V : \bar{v} \in V\}$. The facts below are easily deduced from the definition. They were originally established in [P1] and like most other results from there which we quote here, appear now in [P3].

Fact 2.5: *For U and E as above,*

(i) *If $U = \cup U_i$ for some finite collection of definable sets then*

$$\dim\left(\frac{U}{E}\right) = \max\left\{\dim\left(\frac{U_i}{E}\right)\right\}.$$

(ii) *Let $U \subseteq M^n$ be a definable set and let E_1, E_2 be two definable equivalence relations on U such that $E_1 \subseteq E_2$ and such that for all $\bar{u} \in U$, $\dim\left(\frac{[\bar{u}]_{E_2}}{E_1}\right) \leq r$, then*

$$\dim\left(\frac{U}{E_1}\right) \leq \dim\left(\frac{U}{E_2}\right) + r.$$

(iii) *As in (ii), but replacing $\leq$ by $\geq$ in both places.*

(iv) *Assume that $V \subseteq M^r$ is a definable set and F a definable equivalence relation on V. Let $f : U \longrightarrow V$ be a definable function such that for all $\bar{u}_1, \bar{u}_2 \in U$ we have $\bar{u}_1 E \bar{u}_2$ iff $f(\bar{u}_1) F f(\bar{u}_2)$. Then*

$$\dim\left(\frac{U}{E}\right) = \dim\left(\frac{\mathrm{Im}(f)}{F}\right).$$

(v) *If* $\dim\left(\frac{U}{E}\right) \leq l$ *then for every generic* $\bar{u}_0$ *in* U *over* A *we have* $\dim([\bar{u}_0]_E) \geq \dim(U) - l$.

(vi) *If for all* $\bar{u} \in U$ $\dim([\bar{u}]_E) \geq \dim(U) - l$ *then* $\dim(U/E) \leq l$.

We will make use of one more fact regarding definable equivalence relations.

FACT 2.6: *Assume that* U *and* E *above are* A*-definable. Let* $\bar{u}_0$ *be generic in* U *over* A *and let* $\bar{v}_0$ *be generic over* $\bar{u}_0$ *in* $U_1 = [\bar{u}_0]_E$*. Then* $\bar{u}_0$ *is generic in* U_1 *over* $\bar{v}_0$.

Proof: On one hand, $\dim(\bar{v}_0/\bar{u}_0) = \dim(U_1) \geq \dim(\bar{u}_0/\bar{v}_0)$ (since U_1 is $\bar{v}_0$-definable). On the other hand, $\dim(\bar{v}_0/\bar{u}_0) = \dim(\bar{u}_0\bar{v}_0/\emptyset) - \dim(\bar{u}_0/\emptyset) \leq \dim(\bar{u}_0\bar{v}_0/\emptyset) - \dim(\bar{v}_0/\emptyset) = \dim(\bar{u}_0/\bar{v}_0)$. ∎

2.3 GROUPS AND GROUP-INTERVALS. For the sake of simplicity, we make the convention that whenever we have an o-minimal group it is assumed to be an ordered group. We will make use of the following:

PROPOSITION 2.7 ([PS]): *Let* $\mathcal{M}$ *be an o-minimal group. Then* $\mathcal{M}$ *has no definable subgroups except* $\{0\}$ *and* M*. Hence,* $\mathcal{M}$ *is abelian and divisible.*

As a corollary, if $\mathcal{M}$ is an o-minimal group, then the binary function $+$ is continuous.

Definition 2.8: Let $\mathcal{M} = \langle M; < \ldots \rangle$ be an o-minimal structure with endpoints, say $M = [a, b]$. We say that $\mathcal{M}$ **has a group-interval structure**, or **is a group-interval**, if there is a 0-definable partial binary operation $+$ in the structure satisfying:

1. $+$ is continuous, and strictly increasing in each variable.
2. $+$ is associative, where defined; i.e., $(x + y) + z$ is defined if and only if $x + (y + z)$ is, and then they are equal.
3. $+$ is commutative, when defined.
4. There is an identity 0 in M and for each $x \in M$ an inverse $-x$, and we always have $0 + x = x$ and $x + (-x) = 0$.
5. For $c > 0$ in M, the function $x \mapsto c + x$ has domain $[a, b + (-c)]$ and range $[a + c, b]$.

6. For $c < 0$, the function $x \mapsto c + x$ has domain $[a + (-c), b]$ and range $[a, b + c]$.

The obvious example is a closed interval of an o-minimal ordered group. We will see below (Proposition 5.2) that this is the **only** example thus the name **group-interval** is indeed justified. This definition is not quite identical with the one in [P1]; in particular, there commutativity is derived from the other properties. Also, in [P1] parameters are allowed in the definition of $+$ and $[a, b]$. Our focus here is different; we are not interested in finding a group-interval inside another structure, but rather taking a group-interval (or a group) as our basic structure. We are assuming then that whatever parameters are necessary have already been fixed.

There are a few things to be said about this definition. First, a similar notion could easily have been defined without assuming the structure is o-minimal, but we stick to the o-minimal case. Several properties of o-minimal group-intervals were derived in [P1]; we will have use for three, besides commutativity. First, the operation is clearly torsion-free. A little harder to see is that it is divisible; in particular, for any x in a group-interval, there is a unique element $\frac{1}{2}x$ so that $\frac{1}{2}x + \frac{1}{2}x = x$. Finally, for any x, y in a group-interval, the sum $\frac{1}{2}x + \frac{1}{2}y$ is defined.

We remind the reader here of the first theorem quoted in the introduction; any CF structure with nontrivial geometry has an interval which has a group-interval structure, definable in the original structure.

Notice that there is no restriction on what further structure a group-interval may have. We follow the usual stability-theoretic practice of using the word "group" for any structure with a 0-definable group operation on the universe, with possibly other structure. The same convention applies for "group-interval". Indeed, our plan here is to determine precisely which expansions of a pure ordered group (or group-interval) structure can be o-minimal and satisfy the CF property.

In contrast to the convention for groups, we will use the term "ordered vector space" with almost complete strictness, allowing only expansions by constants. Once and for all, we define an **ordered vector space (o.v.s.)** to be a structure $\mathcal{V} = \langle V; +, <, 0, d, a\colon d \in D, a \in V_0\rangle$ where D is some ordered division ring, V an infinite ordered D-space, and V_0 some collection of distinguished points. (Without loss of generality V_0 is a subspace of V, possibly trivial.) By an **interval in an ordered vector space** we mean a structure $\mathcal{I}$ whose universe is an interval I of V for some o.v.s. $\mathcal{V}$, and this set is endowed with all the structure it inherits from $\mathcal{V}$. We assume that the endpoints of I are in V_0. Let L be the language of $\mathcal{V}$; for every L-formula $\chi(\bar{x})$ with no parameters, we have an $\mathcal{I}$-predicate $P_\chi(\bar{x})$. For $\bar{a} \subseteq I$, we have $\mathcal{I} \models P_\chi(\bar{a})$ if and only if $\mathcal{V} \models \chi(\bar{a})$. The interval in an ordered vector space is then the structure $\langle I; P_\chi \colon \chi \in L\rangle$. We normally assume (with no loss of generality) that for some $c > 0$ in V, $I = [-c, c]$.

For any structure $\mathcal{M}$ and 0-definable $X \subseteq M$, we use the notation $\mathcal{M}|X$ for the structure $\mathcal{X}$ created as $\mathcal{I}$ was created in the last paragraph. That is, the language of $\mathcal{M}|X$ consists of the P_χ's as above, with the obvious interpretation. So in the last paragraph, $\mathcal{I} = \mathcal{V}|I$.

Here is perhaps the appropriate place to mention that for the structure $\mathcal{I}$ just mentioned, we have quantifier elimination in a rather natural language. For $x, y \in I$, define

$$x \oplus y = \frac{x+y}{2}.$$

Also, let D' be the collection of all $d \in D$ so that $d(I) \subseteq I$. So $d \in D'$ if and only if $|d| \leq 1$. (We use the absolute value notation here and elsewhere in the obvious way for any ordered group; $|x|$ is either x or $-x$, whichever is ≥ 0.) In the language L' we take $\oplus$ and the elements of D' as basic function symbols and also add names for all the algebraic points of $\mathcal{I}$. We let $\mathcal{I}'$ be the obvious structure on I for L'. The claim is that $Th(\mathcal{I}')$ eliminates quantifiers and that $\mathcal{I}$ and $\mathcal{I}'$ are interdefinable, which term we will explain very shortly. We leave the q.e. to the reader; it's not much harder than the q.e. result for ordered vector spaces. The interdefinability of the two structures is an easy consequence of Proposition 5.1 below.

For $M_1 \subseteq M_2$ two o-minimal groups or group-intervals we call $a \in M_2$ **an infinitesimal with respect to** M_1 if $0 < |a| < b$ for all $b \neq 0$ from M_1. a is **infinite with respect to** M_1 if $a > b$ for all $b \in M_1$.

In the past few years, the word "reduct" and its relatives have been used for various distinct notions. We therefore spell out exactly how we will use the words "reduct" and "interdefinable".

Definition 2.9: Let $\mathcal{M}$ and $\mathcal{M}'$ be two structures with the same universe M for possibly different languages. We say $\mathcal{M}$ is a **reduct** of $\mathcal{M}'$ if for every n, all the subsets of M^n 0-definable in $\mathcal{M}$ are also 0-definable in $\mathcal{M}'$. Note that we do **not** allow parameters in the definition. If each is a reduct of the other, we say that $\mathcal{M}$ and $\mathcal{M}'$ are **interdefinable**. For two theories T and T', we say that

T **is a reduct of** T' if there are structures $\mathcal{M}$ and $\mathcal{M}'$ such that $T = Th(\mathcal{M})$, $T' = Th(\mathcal{M}')$ and $\mathcal{M}$ is a reduct of $\mathcal{M}'$. Similarly, we define T **is interdefinable with** T'.

3. The CF property and linear theories

Throughout this section we will assume that T is an o-minimal theory. Every structure $\mathcal{M}$ we consider is assumed to be a model of T.

Suppose that $W \subseteq M^{n+1}$ is definable and that $F\colon W \to M$ is a definable function. Let $U \subseteq M^n$ be the projection of W on the first n coordinates. Then F generates a definable family of (partial) unary functions $\mathcal{F} = \{f_{\bar{u}}\colon \bar{u} \in U\}$, defined by $f_{\bar{u}}(x) = F(\bar{u}, x)$. We say that $\mathcal{F}$ **is continuous** if the function F is continuous on its domain. For $\bar{u} \in U$, denote by $I_{\bar{u}}$ the domain of $f_{\bar{u}}$. For $J \subseteq M$ and $\bar{u}_1, \bar{u}_2 \in U$, we say that $f_{\bar{u}_1}|J = f_{\bar{u}_2}|J$ if $I_{\bar{u}_1} \cap J = I_{\bar{u}_2} \cap J$ and for all x in that set we have $f_{\bar{u}_1}(x) = f_{\bar{u}_2}(x)$. $\mathcal{F}$ induces the following two equivalence relations on U:

Definition 3.1: (i) For $J \subseteq M$, define $\bar{u}_1 \sim_J \bar{u}_2$ iff $f_{\bar{u}_1}|J = f_{\bar{u}_2}|J$.
(ii) For $a \in M$, define $\bar{u}_1 \sim_a \bar{u}_2$ iff there exists $a' > a$ such that $\bar{u}_1 \sim_{(a,a')} \bar{u}_2$. ∎

It is immediate that the two are indeed equivalence relations. They are definable with the parameters used to define U, $\mathcal{F}$, J and a. These equivalence relations depend on $\mathcal{F}$ but we use them only when it is clear which $\mathcal{F}$ we refer to. Notice that if $f_{\bar{u}_1}, f_{\bar{u}_2}$ are not defined on any interval of the form (a, a') then $\bar{u}_1 \sim_a \bar{u}_2$.

For $\mathcal{F}$ as above, the relation $\sim_a$ induces an obvious equivalence relation on the $f_{\bar{u}}$'s. We call the equivalence class of $f_{\bar{u}}$ with respect to this relation **the germ of** $f_{\bar{u}}$ **at** a and denote it by $[f_{\bar{u}}]$. By o-minimality, if $f_{\bar{u}_1}, f_{\bar{u}_2}$ are two functions defined on (a, a') then there is $a_1 > a$ such that, on the interval (a, a_1), either $f_{\bar{u}_1} < f_{\bar{u}_2}$ or $f_{\bar{u}_1} > f_{\bar{u}_2}$ or $f_{\bar{u}_1} = f_{\bar{u}_2}$. It is easy to verify that this linear ordering induces a definable linear ordering on the germs at a. In particular, if

$$\dim\left(\frac{U}{\sim_a}\right) = 0$$

then each equivalence class in $\frac{U}{\sim_a}$ is definable with the same parameters that were used to define $\sim_a$.

We can now define the CF property.

Definition 3.2: We say that $\mathcal{M}$ **has the CF (Collapse of Families) property** if for every definable family of functions $\mathcal{F} = \{f_{\bar{u}}: \bar{u} \in U\}$, and every $a \in M$ we have

$$\dim\left(\frac{U}{\sim_a}\right) \leq 1.$$

We also call such an $\mathcal{M}$ **a CF structure.** ∎

As we later show, in the presence of a group or a group-interval the CF property is equivalent to saying that all definable sets are semi-linear.

PROPOSITION 3.3: *If $\mathcal{M}$ is an ω-saturated structure then the following are equivalent:*
(i) $\mathcal{M}$ has the CF property.
(ii)Assume that $\mathcal{F}$ and U as above are A-definable. Let a be in M and assume that $\bar{u}_0$ is generic in U over aA. Then there is a neighborhood $V \subseteq U$ of $\bar{u}_0$ and $a' > a$ such that

$$\dim\left(\frac{V}{\sim_{(a,a')}}\right) \leq 1.$$

The proof is immediate once we establish the following lemma:

LEMMA 3.4: *Let $\mathcal{M}$ be an ω-saturated model. Let U, $\mathcal{F}$ and a be as before and assume that U and $\mathcal{F}$ are A-definable. Then:*
(i) If

$$\dim\left(\frac{U}{\sim_a}\right) \leq l,$$

then for every $\bar{u}_0$ generic in U over aA there is a definable neighborhood $V \subseteq U$ of $\bar{u}_0$ and $a' > a$ such that

$$\dim\left(\frac{V}{\sim_{(a,a')}}\right) \leq l.$$

(ii) Assume that for every A-definable subset $U_0 \subseteq U$ and for every generic $\bar{u}_0 \in U_0$ over aA there is a neighborhood $V \subseteq U_0$ of $\bar{u}_0$ and $a' > a$ such that

$$\dim\left(\frac{V}{\sim_{(a,a')}}\right) \leq l.$$

Then

$$\dim\left(\frac{U}{\sim_a}\right) \leq l.$$

Proof: Without loss of generality $A = \emptyset$ and a is 0-definable. Let $\dim(U) = k$. In the following argument all equivalence classes of the form $[\bar{v}]_E$ are taken in U.

(i) Let $\bar{u}_0$ be generic in U over $\emptyset$ and $\bar{v}_0$ generic in $[\bar{u}_0]_{\sim_a}$ over $\bar{u}_0$. There is then $a_1 > a$ such that $f_{\bar{u}_0}|(a, a_1) = f_{\bar{v}_0}|(a, a_1)$. Clearly, if $a_2 \in (a, a_1)$ then we still have $f_{\bar{u}_0}|(a, a_2) = f_{\bar{v}_0}|(a, a_2)$. So without loss of generality, $a_1 \notin dcl(\bar{u}_0\bar{v}_0)$. By Fact 2.6, $\bar{u}_0$ is generic in $[\bar{u}_0]_{\sim_a}$ over $\bar{v}_0$, hence it is also generic there over $a_1\bar{v}_0$. It follows that there is a neighborhood $V_1 \subseteq [\bar{u}_0]_{\sim_a}$ of $\bar{u}_0$ such that for every $\bar{v} \in V_1$ the formula $f_{\bar{v}}|(a, a_1) = f_{\bar{v}_0}|(a, a_1)$ holds. In particular, this implies that $V_1 \subseteq [\bar{u}_0]_{\sim_{(a,a_1)}}$ and therefore $\dim([\bar{u}_0]_{\sim_{(a,a_1)}}) = \dim([\bar{u}_0]_{\sim_a}) \geq k - l$ (the latter by Fact 2.5 (v)).

Pick $a' \in (a, a_1)$, not in $dcl(\bar{u}_0)$. Since $[\bar{u}_0]_{\sim_{(a,a_1)}} \subseteq [\bar{u}_0]_{\sim_{(a,a')}} \subseteq [\bar{u}_0]_{\sim_a}$ we still have $\dim([\bar{u}_0]_{\sim_{(a,a')}}) \geq k - l$. Let $\varphi(\bar{u}_0, a')$ be the formula saying '$\dim([\bar{u}_0]_{\sim_{(a,a')}}) \geq k - l$' and let $V = \{\bar{v} \in U : \mathcal{M} \models \varphi(\bar{v}, a')\}$. $\bar{u}_0$ is in V and since it is generic in U over a' we have $\dim(V) = \dim(U)$. By Fact 2.5 (vi), we have

$$\dim\left(\frac{V}{\sim_{(a,a')}}\right) \leq l.$$

(ii) We will use induction on $\dim(U)$. Let $\bar{u}_0$ be generic in U over $\emptyset$. There is a neighborhood $V \subseteq U$ of $\bar{u}_0$ and $a' > a$ such that

$$\dim\left(\frac{V}{\sim_{(a,a')}}\right) \leq k - l.$$

But then, by Fact 2.5 (v), $\dim([\bar{u}_0]_{\sim_{(a,a')}} \cap V) \geq k - l$, which implies that $\dim([\bar{u}_0]_{\sim_a}) \geq k - l$.

Let $U_1 = \{\bar{u} \in U : \dim([\bar{u}]_{\sim_a}) \geq k - l\}$. Then

$$\dim\left(\frac{U_1}{\sim_a}\right) \leq l.$$

The above argument shows that $\dim(U \setminus U_1) < k$ hence, by applying induction to $U \setminus U_1$ (U_1 still satisfies the hypothesis of (ii)) we have

$$\dim\left(\frac{U \setminus U_1}{\sim_a}\right) \leq l.$$

But, by Fact 2.5 (i),

$$\dim\left(\frac{U}{\sim_a}\right) = \max\{\dim\left(\frac{U_1}{\sim_a}\right), \dim\left(\frac{U \setminus U_1}{\sim_a}\right)\} \leq l. \qquad \blacksquare$$

Using the equivalence in Proposition 3.3 we can now rephrase a result from [P1].

PROPOSITION 3.5: *If $\mathcal{M}$ is a locally modular structure then it has the CF property.*

Remarks:

1. The standard example of a CF structure is an ordered vector space. This structure is also modular. However, as Example 4.5 shows, $\mathcal{M}$ may have the CF property and not even be locally modular. A standard example where the CF property fails is a real closed field, where the family of curves given by $\{f_{u_1u_2}(x) = u_1x + u_2\colon u_1, u_2 \in \mathbb{R}\}$ doesn't collapse at any point in $\mathbb{R}$.
2. If $\mathcal{M}$ is a CF structure and $\mathcal{N} \equiv \mathcal{M}$ then $\mathcal{N}$ is also a CF structure (the CF property clearly can be written as a first order scheme). We call a theory T a **CF theory** if it has a CF model. Note that the CF property is preserved under reducts, hence the following converse of Theorem 1.3 holds: If $\mathcal{M}$ is an ordered reduct of (an interval in) an ordered vector space then $\mathcal{M}$ has the CF property.
3. We should point out that one can define another equivalence relation E_a similar to $\sim_a$, but instead of considering $a' > a$ we could have considered $a' < a$. One can show that $\mathcal{M}$ is a CF structure iff for every $\mathcal{F}$, U and a we have

$$\dim\left(\frac{U}{E_a}\right) \leq 1.$$

We can also define another equivalence relation, call it $\approx_a$, by: $\bar{u}_1 \approx_a \bar{u}_2$ iff there are $a_1 < a < a_2$ such that $f_{\bar{u}_1}|(a_1, a_2) = f_{\bar{u}_2}|(a_1, a_2)$. One can show that $\mathcal{M}$ is a CF structure iff for every generic $a \in M$ we have

$$\dim\left(\frac{U}{\approx_a}\right) \leq 1.$$

Definition 3.6: For $A \subseteq M$, a point $a \in M$ is called **nontrivial over A** if there is an A-definable partial function $h(x, y)$ such that h is defined, continuous and strictly monotone in both variables on some A-definable neighborhood of the point (a, a). a is called **nontrivial** if it is nontrivial over some $A \subseteq M$. ∎

If $f(x)$ is a definable function and $a \in M$, then $\lim_{x\to a^+} f(x)$ (the limit from the right of $f(x)$) is always defined and is in $M \cup \{\pm\infty\}$. For $\mathcal{F} = \{f_{\bar{u}}\colon \bar{u} \in U\}$ a definable family of functions and for $a \in M$, $b \in M \cup \{\pm\infty\}$, we define $U_{ab} = \{\bar{u} \in U\colon \lim_{x\to a^+} f(x) = b\}$.

PROPOSITION 3.7: (i) *Assume that $\mathcal{M}$ has the CF property and that a is a point in M, nontrivial over some set $A \subseteq M$. Suppose that $\mathcal{F} = \{f_{\bar{u}}: \bar{u} \in U\}$ is an A-definable continuous family of functions, each defined on some interval $(a, a'(\bar{u}))$. Then, for every $b \in M \cup \{\pm\infty\}$ we have*

$$\dim\left(\frac{U_{ab}}{\sim_a}\right) = 0.$$

(ii) *Assume that $\mathcal{M}$ does not have the CF property, then there is a definable $\mathcal{F}$, as above, and a point $a \in M$, such that*

$$\dim\left(\frac{U_{aa}}{\sim_a}\right) \geq 1.$$

Proof: (i) Let $h(x, y)$ be strictly monotone in both variables on some neighborhood of the point (a, a) and let $\mathcal{F}$, U and b be as above. By restricting ourselves to U_{ab} we will assume that $U = U_{ab}$. We may assume that there is $\bar{u}_0 \in U$ such that $[f_{\bar{u}_0}]$ is not the germ of a constant function (for otherwise we are done). Replacing $\mathcal{F}$ by the family $\{f_{\bar{u}_0}^{-1} f_{\bar{u}}: \bar{u} \in U\}$ (where $f_{\bar{u}_0}^{-1} f_{\bar{u}}$ is taken where defined, i.e. on some interval near a), we may assume that $a = b$. We need to show that $\mathcal{F}$ has only finitely many germs at a. Assume that there are infinitely many such germs, all "going through" the point (a, a) (see Fig. I).

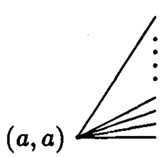

Fig. I

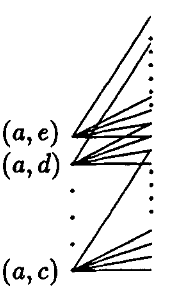

Fig. II

Consider the following family of functions:

$$g_{\bar{u}t}(x) = h(t, f_{\bar{u}}(x)).$$

By the continuity of $\mathcal{F}$, there is a neighborhood $V \subseteq U$ of $\bar{u}_0$ and $a' > a$ such that $g_{\bar{v}t}(x)$ is defined and continuous for all $\bar{v} \in V$, $t \in (a, a')$ and $x \in (a, a')$. Let

$I = (a, a')$ and $\mathcal{G} = \{g_{\bar{v}t}\colon \bar{v} \in V, t \in I\}$. The monotonicity of h in both variables ensures that $\mathcal{G}$ is a union of infinitely many translates of $\mathcal{F}$. The germs in any two such translates "go through" different points $(a, c), (a, d)$ (see Fig. II). Using the properties of quotients from Fact 2.5, one can show that

$$\dim\left(\frac{V \times I}{\sim_a}\right) \geq 2$$

(with respect to $\mathcal{G}$), contradicting the CF property. We leave the details to the reader.

(ii) Let $\mathcal{F} = \{f_{\bar{u}}\colon \bar{u} \in U\}$and $a \in M$ witness the failure of the CF property. Namely,

$$\dim\left(\frac{U}{\sim_a}\right) \geq 2.$$

Notice that if there is $b \in M \cup \{\pm\infty\}$ such that

$$\dim\left(\frac{U_{ab}}{\sim_a}\right) \geq 1$$

then, as in part (i), we may assume $b = a$ and we will be done. Let U_1 be the union of all the U_{ab}'s, $b \in M \cup \{\pm\infty\}$. It is easy to verify that $\dim\left(\frac{U}{\sim_a}\right) = \dim\left(\frac{U_1}{\sim_a}\right)$. For $\bar{u}_1, \bar{u}_2 \in U_1$, define $\bar{u}_1 E \bar{u}_2$ iff $\bar{u}_1$ and $\bar{u}_2$ are in the same U_{ab}. It is easy to see that

$$\dim\left(\frac{U_1}{E}\right) = 1$$

and by Fact 2.5 (ii), there is $b \in M \cup \{\pm\infty\}$ such that

$$\dim\left(\frac{U_{ab}}{\sim_a}\right) \geq 1. \qquad \blacksquare$$

Assume now that $\mathcal{M} = \langle M; +, <, \cdots \rangle$ is an o-minimal group, or a group-interval. First we make the following definition.

Definition 3.8: For $\mathcal{M}$ as above, let f be a definable partial function from a definable set $U \subseteq M^n$ into M.

(i) We say that f **is linear on** U if the following hold:

For $\mathcal{M}$ a group, given $\bar{x}_1, \bar{x}_2 \in U$ and $\bar{t} \in M^n$,

$$\text{if } \bar{x}_1 + \bar{t}, \bar{x}_2 + \bar{t} \in U \text{ then } f(\bar{x}_1 + \bar{t}) - f(\bar{x}_1) = f(\bar{x}_2 + \bar{t}) - f(\bar{x}_2).$$

For $\mathcal{M}$ a group-interval, given $\bar{x}_1, \bar{x}_2 \in U$ and $\bar{t} \in M^n$,

$$\text{if } \bar{x}_1/2 + \bar{t}/2, \bar{x}_2/2 + \bar{t}/2 \in U$$
$$\text{then } f(\bar{x}_1/2 + \bar{t}/2)/2 - f(\bar{x}_1/2)/2 = f(\bar{x}_2/2 + \bar{t}/2)/2 - f(\bar{x}_2/2)/2.$$

(We divide by 2 to ensure that + and − are defined.)
(ii) We say that f **is piecewise linear on** U if we can partition U into definable sets $U_1, \dots, U_k$ such that f is linear on each U_j.

PROPOSITION 3.9: *Assume that $\mathcal{M}$ has the CF property. If $f(x)$ is a definable function from an interval $(a,b) \subseteq M$ into M then f is piecewise linear on (a,b).*

Proof: We prove the result for a group $\mathcal{M}$. The proof for a group-interval is similar.

Without loss of generality we may assume that f is continuous on (a,b). Define the following family of partial functions parametrized over (a,b). For $x \in (a,b)$ and $t > 0$, let

$$g_x(t) = f(x+t) - f(x).$$

Notice that every g_x is defined on some neighborhood of 0 and that for all x we have $g_x(0) = 0$. The point 0 is a nontrivial point over $\emptyset$ since the function $x+y$ is defined in some neighborhood of $(0,0)$.

Now, using Lemma 3.7, we know that

$$\dim\left(\frac{(a,b)}{\sim_0}\right) = 0.$$

So, by partitioning (a,b) into finitely many sets we may assume that $\frac{(a,b)}{\sim_0}$ contains a single class. Namely, given any $x_1, x_2 \in (a,b)$,

$$\exists \epsilon > 0 \forall t \in [0,\epsilon], (g_{x_1}(t) = g_{x_2}(t)),$$

which means

$$\exists \epsilon > 0 \forall t \in [0,\epsilon](f(x_1+t) - f(x_1) = f(x_2+t) - f(x_2)). \tag{1}$$

To conclude the proof we want to omit the ϵ-restriction in the above statement. Given any $x_1, x_2 \in (a,b)$, let $m = \min\{b - x_1, b - x_2\}$. Define

$$L_{x_1,x_2} = \{\epsilon \in (0,m) \colon \forall t, 0 < t < \epsilon, f(x_1+t) - f(x_1) = f(x_2+t) - f(x_2)\}.$$

It is sufficient to show that $L_{x_1,x_2} = (0, m)$.

By (1), $L_{x_1,x_2} \neq \emptyset$. By the definition of L_{x_1,x_2} and the continuity of f, it is clearly a closed set. To see that it is open, assume that $\epsilon \in L_{x_1,x_2}$. Since f is continuous we have $f(x_1+\epsilon)-f(x_1) = f(x_2+\epsilon)-f(x_2)$. By (1), there is an ϵ_1 such that for all $0 < t < \epsilon_1$ we have $f(x_1+\epsilon+t)-f(x_1+\epsilon) = f(x_2+\epsilon+t)-f(x_2+\epsilon)$.

Putting the two together we get

$$f(x_1 + \epsilon + t) - f(x_1) = f(x_2 + \epsilon + t) - f(x_2),$$

which implies that $(0, \epsilon + \epsilon_1) \subseteq L_{x_1,x_2}$.

So, L_{x_1,x_2} is a clopen set but since every interval is definably connected we have $L_{x_1,x_2} = (0, m)$. ∎

PROPOSITION 3.10: *Let $(a, b) \subseteq M$ be an interval containing 0 and assume that $f\colon (a,b) \to M$ is a $\bar{c}$-definable function, linear on (a,b), for which $f(0) = 0$. Then there is a 0-definable function g and a $\bar{c}$-definable interval $(a_1, b_1) \subseteq (a, b)$ containing 0 such that $f|(a_1, b_1) = g|(a_1, b_1)$.*

Proof: Assume that $\bar{c}$ is in M^n. Let $\varphi(x, y, \bar{c})$ be the formula defining the graph of f. The above definition of linearity makes it clear that this is a first order notion in the language of $\mathcal{M}$. We may assume then that for every $\bar{u}$, the formula $\varphi(x, y, \bar{u})$ defines the graph of a linear function $f_{\bar{u}}$ on an interval containing 0 and that $f(0) = 0$.

Consider the family of functions $\{f_{\bar{u}} : \bar{u} \in M^n\}$. By Lemma 3.7 we know that

$$\dim\left(\frac{M^n}{\sim_0}\right) = 0.$$

Hence, as we have already commented, $[\bar{c}]_{\sim_0}$ is 0-definable. But since every function in $[f_{\bar{c}}]$ is linear and agrees with $f_{\bar{c}}$ on some interval containing 0 it must agree, by o-minimality, everywhere on their common domain. Take now g to be the union of all functions in $[f_{\bar{c}}]$. g is 0-definable and we can define (a_1, b_1) to be the interval on which g agrees with $f_{\bar{c}}$. This interval is clearly $\bar{c}$-definable. ∎

Definition 3.11: Let $\hat{T}$ be an o-minimal theory of a group or a group-interval. We say that $\hat{T}$ **is linear** if Proposition 3.9 and Proposition 3.10 hold when we replace $\mathcal{M}$ by any model of $\hat{T}$. ∎

As we have just shown, if $\hat{T}$ has the CF property then it is linear. Our work below implies that the converse is also true (one may also verify it directly).

4. Partial endomorphisms and the moduar case

We fix, for the purposes of this section, an o-minimal theory T so that any $\mathcal{M} \models T$ is either a group or a group-interval, with operation $+$. We call a 0-definable partial unary function f of $\mathcal{M} \models T$ a **partial endomorphism (p.e.)** if:

1. The domain of f, $dom(f)$, is either all of M or $(-c, c)$ for some $c > 0$;
2. Whenever a, b and $a+b$ are all in $dom(f)$ (so in particular $a+b$ is defined) then $f(a+b) = f(a) + f(b)$.

Notice that because $+$ is continuous with respect to $<$, any partial endomorphism is, too. Obviously, the fact that a formula of two variables defines the graph of a p.e. is independent of the particular model $\mathcal{M}$ chosen.

It is clear that if f and g are p.e.'s, then $f+g$, $f \circ g$, $-f$ and f^{-1} (if $f \neq 0$ on its domain) are also p.e.'s. Here for example, the domain of $f+g$ is $dom(f) \cap dom(g)$. We define an equivalence relation E on p.e.'s by fEg if and only if for all $a \in dom(f) \cap dom(g)$, we have $f(a) = g(a)$.

PROPOSITION 4.1: *Suppose f and g are p.e.'s.*
(1) *If there is some $a > 0$ in $dom(f) \cap dom(g)$ such that $f(a) > g(a)$, then for any $b > 0$ in the common domain, $f(b) > g(b)$.*
(2) *If $f(a) = g(a)$ for some $a \neq 0$ in their common domain, fEg.*

Proof: (1) By the assumption and the continuity of f and g, the set $\{x > 0\colon f(x) > g(x)\}$ is open and nonempty in $dom(f) \cap dom(g)$. If its infimum c is positive, choose any c' in the set so that $c' < 2c$; we have $f(c') > g(c')$ but $f(\frac{c'}{2}) \leq g(\frac{c'}{2})$, a contradiction. Thus $c = 0$. Now if this set is not all the positive elements of $dom(f) \cap dom(g)$, let d be the supremum of the first interval it contains. Consider $f(\frac{d}{2})$ and $g(\frac{d}{2})$. Thus $f(b) > g(b)$ for all $b > 0$ in the common domain. For (2), notice that if it is not the case that fEg, we have $f(a) \neq g(a)$ for some nonzero a in $dom(f) \cap dom(g)$, Without loss of generality, $f(a) > g(a)$ and $a > 0$. By (1), $f(b) > g(b)$ for all $b > 0$ in the common domain, so $f(b) < g(b)$ for all $b < 0$ in the common domain. This gives (2). ∎

It is now clear that E is a congruence for the operations mentioned in the paragraph preceding the proposition. Further, the E-classes are naturally ordered: $f/E < g/E$ if and only if for some (any) $a > 0$ in $dom(f) \cap dom(g)$ we have $f(a) < g(a)$. The proposition makes it clear that the choice of f, g in their respective classes and $a > 0$ is irrelevant. Thus the collection of E-classes naturally forms an ordered division ring, which we will call D. Also, since any definable partial function in any o-minimal structure is piecewise monotone, an easy consequence of this proposition is that any p.e. not constantly 0 on its domain is either order-preserving or order-reversing (depending on whether its E-class is > 0 or < 0).

If T is linear then, for any $\bar{a} \subseteq M$, every p.e. 0-definable in $Th(\mathcal{M}, \bar{a})$ is the restriction of a p.e. 0-definable in $\mathcal{M}$ to an $\bar{a}$-definable set. Also, if $a \in acl(b)$ for $a, b \in \mathcal{M} \models T$ there is a 0-definable linear partial function g so that $a = g(b)$. We may choose g which has domain (u, v) and range (u', v'), where u, v, u', v' are all algebraic points. If $0 \in (u, v)$, let $e = 0$ and $e' = g(0)$; if $u \geq 0$, let $e = u$ and $e' = u'$, and if $v \leq 0$, let $e = v$ and $e' = v'$. In any case, we see that by translating g to the origin and then extending it in an obvious fashion we have that there is a p.e.f such that $a = f(b + e) + e'$, where e and e' are algebraic points. It is clear also that if T is linear, $\mathcal{M} \models T$ and $\bar{b} \subseteq M$, then $Th(\mathcal{M}, \bar{b})$ is also linear. Thus for any $c \in acl(a, \bar{b})$ in $\mathcal{M}$, we have that $c = f(a + b_1) + b_2$, where each of b_1, b_2 is algebraic in $\bar{b}$ and f is some p.e. This will be used in the following.

PROPOSITION 4.2: *Suppose that T is a linear o-minimal theory. Let $\mathcal{M}'$ be the reduct of $\mathcal{M} \models T$ to the language L' containing $+$, $<$, names for the elements of $acl(\varphi) \subseteq \mathcal{M}$, and a symbol for each (0-definable) p.e. of $\mathcal{M}$. Then $\mathcal{M}$ and $\mathcal{M}'$ are interdefinable.*

Proof: We show by induction on n that any subset of M^n 0-definable in $\mathcal{M}$ is in fact 0-definable in $\mathcal{M}'$. This is clear for $n = 1$, and by Corollary 2.2, we need only check that any partial function h from M^n to M 0-definable in $\mathcal{M}$ is 0-definable in $\mathcal{M}'$. By induction, we may assume that the domain of h is 0-definable in $\mathcal{M}'$.

Fix $\mathcal{N} \models T$ which is $|T|^+$-saturated and suppose that $a, \bar{b}$ is in $dom(h)$ taken in $\mathcal{N}$. As noted above, for $c = h(a, \bar{b})$ we have a p.e.f and $b_1, b_2 \in acl(\bar{b})$ so that $c = f(a + b_1) + b_2$. f is in L'. Write $b_i = g_i(\bar{b})$, where each g_i is by induction 0-definable in $\mathcal{M}'$. Again by induction, for each such triple f, g_1, g_2 the set $\{(x,\bar{y})\colon h(x,\bar{y}) = f(x+g_1(\bar{y})) + g_2(\bar{y})\}$ is 0-definable in $\mathcal{M}'$. By compactness the domain of h is the union of finitely many such sets. ∎

The above proof makes it clear that in any model of a linear theory, any definable function of any number of variables is piecewise linear.

Our next project is the following:

LEMMA 4.3: *Fix any $\mathcal{M} \models T$ as above, and suppose that T is locally modular. Let f be any p.e. Then there is a partial endomorphism g with domain all of M such that either gEf or gEf^{-1}. If $\mathcal{M}$ is a group, then we can always choose gEf with domain all of M.*

Proof: By naming some points, we may assume that T is modular. Notice that by the linearity of T and by 4.1(2), we don't get any new p.e.'s whose domain is all of M when we name these points. Let us call a p.e. **total** if its domain is all of M and (for $M = [-a,a]$ a group-interval) **near-total** if its domain includes $[-\frac{a}{n}, \frac{a}{n}]$ for some natural number n. In case $\mathcal{M}$ is a group, we use near-total as a synonym for total. Consider the set

$$\mathcal{F} = \{f'\colon f' \text{ a p.e. which is not near-total}\}$$

and for any $\mathcal{N} \models T$, the partial type

$$p_N = \{x + e \text{ is either undefined or not in } dom(f')\colon f' \in \mathcal{F}, e \in N\}.$$

(In case our structure is a group, any element infinite with respect to N satisfies p_N.) We claim that this type is consistent. Indeed, if it isn't we have $e_1, \dots e_k \in N$ and $f_1, \dots f_k$ p.e.'s in $\mathcal{F}$ so that every element of N is in one of $-e_i + dom(f_i)$. We can suppose that f_1 has the largest domain of all the f_i's; but then finitely many translates of $dom(f_1)$ cover all of N, clearly contradicting that f_1 is not near-total.

For our given model $\mathcal{M}$, choose any $b \models p_M$ and then c independent (with respect to the algebraic closure) from b over M such that $b - c$ is infinitesimal with respect to M; in particular, $b-c$ is in $dom(f)$ for any p.e.f and also $c \models p_M$. Notice also that $f(b-c)$ is infinitesimal with respect to M. Choose any $\mathcal{N} \models T$ containing $M \cup \{b,c\}$ and then d realizing p_N. Now let $e = f(b-c) + d$; it is immediate that $e \models p_N$.

Now by the modularity of any model containing all of our points, we have some element $u \in acl\{b,d\} \cap acl\{c,e\}$ not in $acl(\varphi)$. So there are p.e.'s h and k with

$u = h(d + b_1) + b_2 = k(e + c_1) + c_2$, where b_1, b_2 are algebraic in b and c_1, c_2 in c. We must have both h and k near-total, and we may assume they are in fact total. To see this, notice that the functions naturally denoted by $h(\frac{x}{n})$ and $k(\frac{x}{n})$ are total for suitable n; replace h and k by these, and then replace u, b_2 and c_2 by their nth parts. Thus $u = (h(d) + h(b_1)) + b_2$, and letting $b' = h(b_1) + b_2$ (which is defined) we have $u = h(d) + b'$ and $b' \in acl(b)$. Similarly, there is $c' \in acl(c)$ so that $u = k(e) + c'$.

It is easily seen that $h = k$. For we have now $h(d) + b' = k(f(b - c) + d) + c'$. Since d is independent from b, c there is an interval I around d such that the latter equation holds with any $d' \in I$ replacing d. In particular, by taking $\epsilon > 0$ small enough we have $h(d + \epsilon) + b' = k(f(b - c) + d + \epsilon) + c'$, and hence $h(d) + h(\epsilon) = k(f(b-c)+d)+k(\epsilon)+c'$ (h and k are defined at ϵ). It follows that $h(\epsilon) = k(\epsilon)$ and hence $h = k$. Now we have $hf(b-c) = b' - c' = (g_1(b+e_1)+e_2)-(g_2(c+e_3)+e_4)$ where g_1 and g_2 are p.e.'s and the e_i's are algebraic. We must have each g_i near-total, and by replacing them and h as above, we may assume they are total. So we have for algebraic e', e'' that $hf(b - c) = (g_1(b) + e') - (g_2(c) + e'')$.

Since b is independent from c (over $\emptyset$) we may take $\epsilon > 0$ small enough and get, as above, $hf(b - c) + hf(\epsilon) = (g_1(b) + g_1(\epsilon) + e') - (g_2(c) + e'')$. It follows that $hf(\epsilon) = g_1(\epsilon)$, hence $hfEg_1$. Similarly, $hfEg_2$ and in particular $g_1 = g_2$.

We now assume (as we may) that f is not equivalent to 0; we have that $fEh^{-1}g_1$ and $f^{-1}Eg_1^{-1}h$. If the range of g_1 is contained in the range of h, which is of course the domain of h^{-1}, we set $g = h^{-1}g_1$; otherwise we set $g = g_1^{-1}h$. We have g total and either fEg or $f^{-1}Eg$. We may as well notice here that for $\mathcal{M}$ a group-interval, we can have g total only if $-1 \leq g/E \leq 1$ in the division ring D. For $\mathcal{M}$ a group, on the other hand, if $g \neq 0$ is total, then so is g^{-1}, because its domain is the range of g, which is a nonzero subgroup of M. But the only definable subgroups of M are $\{0\}$ and M (Proposition 2.7). ∎

We have more work to do to finish our description of modular group-intervals, but at this point we can easily describe all groups whose theory is o-minimal and has only modular models.

THEOREM 4.4: *Suppose T is o-minimal, modular, and that $\mathcal{M} \models T$ is a group. Then $\mathcal{M}$ with its full structure is interdefinable with an ordered vector space.*

Proof: We set D to be the set of all 0-definable additive endomorphisms of $\mathcal{M}$. Since every p.e. is the restriction of a total endomorphism to some 0-definable interval, they are all 0-definable in the given structure. So this D is essentially the same as the D mentioned above. The result is now immediate from Proposition 4.2. ∎

Example 4.5: Let $\mathcal{R}$ be the structure $(\mathbb{R}; +, <, f)$, where f is the partial function defined on $(-1, 1)$ such that $f(x) = \pi x$ for every x in its domain. $Th(\mathcal{R})$ has models which are not locally modular, as the argument for the lemma demonstrates. Of course $\mathcal{R}$ itself has modular dependence relation, as its algebraic closure operation is the same as that of $\mathcal{R}' = (\mathbb{R}; +, <, g)$, where $g(x) = \pi x$ for all $x \in \mathbb{R}$. $g|(-n, n)$ is easily seen to be definable in $Th(\mathcal{R})$ for any natural number n, but not in a uniform manner. This example indicates that a reduct of a structure whose theory is o-minimal and modular need not be (even locally) modular. $Th(\mathcal{R})$ is easily seen to be linear.

5. Embedding a group-interval in a group

In this section, we will continue our investigation of o-minimal linear theories of group-intervals. But first, we prove a quite general result about ordered vector spaces with extra structure on some bounded subset. We set up some notation first. Let $\langle V; +, <, d\colon d \in D\rangle$ be an ordered vector space over D and $I = [-a_0, a_0]$ be a closed interval of V. Let $\mathcal{P}$ be any collection of subsets of I^n, for various n, containing all those a_0-definable in the vector space structure. For simplicity assume that any predicate 0-definable in $\langle I; P\colon P \in \mathcal{P}\rangle$ is already in $\mathcal{P}$. We use $\mathcal{V}$ to abbreviate the structure $(V; +, <, d, a_0, P\colon d \in D, P \in \mathcal{P})$ and for any $\mathcal{V}^* \succeq \mathcal{V}$ set $I^* = [-a_0, a_0]$ evaluated in $\mathcal{V}^*$.

PROPOSITION 5.1: *We use the notation of the previous paragraph.*

1. *$Th(\mathcal{V})$ admits elimination of quantifiers.*
2. *Let $\chi(\bar{x}, \bar{y})$ be any $\mathcal{V}$-formula and $\bar{a} \subseteq \mathcal{V}^* \succeq \mathcal{V}$. Then there is $\bar{b} \subseteq I^*$ in the vector space closure of $\bar{a}$ and $P(\bar{x}, \bar{z}) \in \mathcal{P}$ so that $\chi(\mathcal{V}^*, \bar{a}) \cap (I^*)^m = P(\mathcal{V}^*, \bar{b})$.*

Proof: First notice that by replacing the formula χ, we may assume that $\bar{a} = (a_1, \ldots, a_k, a_{k+1}, \ldots, a_n)$ where $a_1, \ldots, a_k \in I^*$ and $a_{k+1}, \ldots, a_n$ are linearly independent over I^* in the D-space structure. To see this, we use the fact if b is in the D-closure of $\{a_1, \ldots, a_i\}$ and this set is minimal such, then a_i and b are definable from each other over the previous a_j's; whenever possible, we apply this to replace any a_i by an element of I^*. Having done this, our $\bar{b}$ will be $(a_1, \ldots, a_k)$.

Now any automorphism of $\langle I^*; P\colon P \in \mathcal{P}\rangle$ extends uniquely to a D-space automorphism of the D-closure of I^*; it necessarily preserves the order. Because this closure is convex in V^*, we can extend this to an ordered D-space automorphism of V^* which fixes $a_{k+1}, \ldots a_n$. This gives us that if $\bar{c}$ is another sequence in V^* with the same quantifier-free type as $\bar{a}$, there is an automorphism of V^* taking $\bar{a}$ to $\bar{c}$. With this observation, (1) easily follows.

To see (2), if the conclusion were false, there would be $\mathcal{V}^{**} \succeq \mathcal{V}^*$ having an automorphism which fixes $\bar{b}$ but moves the set $\chi(\mathcal{V}^{**}, \bar{a}) \cap (I^{**})^m$. We restrict this to I^{**} and extend this restricted function to an ordered D-space automorphism of $\mathcal{V}^{**}$ fixing $a_{k+1}, \ldots, a_n$. This is then a full automorphism of $\mathcal{V}^{**}$ fixing $\bar{a}$ but moving $\chi(\mathcal{V}^{**})$, obviously a contradiction. ∎

The above result clearly implies that if we start with an ordered vector space $\mathcal{V}$ and enrich the structure by additional predicates on a bounded subset B, then anything definable in the new structure is in fact definable in a quantifier-free fashion using (perhaps other) predicates on B and the ordered vector space terms. Special cases of this have appeared in [PSS] and [P1].

Our next project is to embed a group-interval $\mathcal{M}$ in a group which reflects many of its properties. Very little of the strength of o-minimality is used in the construction, but here we will only consider the o-minimal case. We recall the facts that $\mathcal{M}$ is abelian (where the operation is defined), that every element has a unique "half", and that for any a, b in M, $\frac{a}{2} + \frac{b}{2}$ is defined.

We create a group $\langle A; +, \ldots\rangle$ containing M in the following fashion: Let $A = (M \times \omega)/E$ where $(a,n)E(b,m)$ if and only if $\frac{a}{2^n} = \frac{b}{2^m}$. (We then have $(a,n)E(\frac{a}{2}, n+1)E(\frac{a}{4}, n+2)\ldots$.) For $(a,n),(b,m) \in M \times \omega$ with $n \geq m$ there is a unique b' with $(b,m)E(b',n)$ and we set $(a,n)/E + (b,m)/E = ((\frac{a}{2} + \frac{b'}{2}), n+1)/E$. The fact that $+$ on M is commutative when defined ensures both that this is well-defined and that it equals $(a+b', n)$ when the latter is defined. It is readily checked that this endows A with an abelian group structure and that $a \mapsto (a,0)/E$ is an embedding of M into A. We will henceforth identify M with its image under this map.

Since $\langle M; +, <\rangle$ is a group-interval it is divisible, and $+$ is continuous with respect to $<$. It is easily seen that $\langle A; +\rangle$ will be divisible and if we order it in the obvious way ($(a,n)/E < (b,n)/E$ if and only if $a < b$ in $\mathcal{M}$), $+$ remains continuous with respect to $<$. We let $\mathcal{P}$ be the collection of all 0-definable predicates of any number of variables in $\mathcal{M}$, and consider the structure

$$\langle A; +, 0, q, <, P, a_0 : P \in \mathcal{P}, q \in Q\rangle.$$

Here we regard a rational $q \in Q$ as a 0-definable (in $\langle A; +\rangle$) function on A, and a_0 is the upper endpoint of M inside A. It is clear that M is a closed interval in A. The following summarizes our construction and demonstrates that several relevant properties pass from a group-interval to the group created from it.

PROPOSITION 5.2: *Let $\mathcal{M} = \langle M; +, <, \ldots\rangle$ be a group-interval. Then there is a divisible ordered abelian group $\mathcal{A} = \langle A; +, <, \ldots\rangle$ such that M is a closed interval of A and the structures $\mathcal{M}$ and $\mathcal{A}|M$ are interdefinable. $\mathcal{A}$ is o-minimal. Further, if $\mathcal{M}$ is a CF structure, so is $\mathcal{A}$.*

Proof: Let $\mathcal{A}$ be the structure described above. It is clear that $\mathcal{M}$ is a reduct of $\mathcal{A}|M$. The converse is an immediate consequence of Proposition 5.1 with $Q = D$.

We now check that $\mathcal{A}$ is o-minimal. By the quantifier elimination result above (Proposition 5.1), we need only check that any atomic formula with parameters from A and a single free variable defines a finite union of points and intervals. We may assume (as $A = \cup_{n\in\omega} nM$) that the parameters and constants $\bar{a}$ come from M. The atomic formulas in one variable x with parameters (including constants) $\bar{a}$ are equations and inequalities of terms in the pure Q-space language and formulas of the form $P(t_1(x,\bar{a}), \ldots, t_k(x,\bar{a}))$, where P is 0-definable in $\mathcal{M}$ and the t_i's are terms of the pure Q-space language.

The intersection of the set defined by such a formula and M is a union of intervals and points, and the same is true for nM for any $n \in \omega$ (since $\mathcal{A}|M$ and $\mathcal{A}|nM$ are isomorphic via the map $x \mapsto nx$). But there is a natural number n such that the given formula $(P(t_1, \ldots, t_k))$ implies $-na_0 \le x \le na_0$ because for all i, we have $t_i(x,\bar{a}) = q_i x + a_i$ where a_i is in the Q-space closure of $\bar{a}$ and $q_i \in Q$, and for at least one i $q_i \neq 0$.

To verify that $\mathcal{A}$ has the CF property, it is easiest to use the characterization in Proposition 3.7. If $\mathcal{A}$ is not a CF structure then there is a family $\mathcal{F} = \{f_{\bar{u}} : \bar{u} \in U\}$and a point $a \in A$ such that

$$\dim\left(\frac{U_{aa}}{\sim_a}\right) \ge 1.$$

But now, we can use $+$ to transfer a to a point in the interior of M. Since every (interior) point of a group (or a group-interval) is nontrivial we get a contradiction to the fact that $\mathcal{M}$ is a CF structure. ∎

6. Embedding a group in a vector space

Our purpose here is to embed a group $\mathcal{A}$ whose theory is o-minimal and linear into an ordered vector space in a natural way. More specifically, we will prove the following:

THEOREM 6.1: *Suppose that $\mathcal{A}$ is an o-minimal abelian group with linear theory. Then there is an elementary extension $\mathcal{V}'$ of $\mathcal{A}$ which is a reduct of $\mathcal{V}$, an ordered vector space.*

This result will lead quickly to a characterization of modular and linear group-intervals. We first prove an effective quantifier-elimination result for $Th(\mathcal{A})$ in an appropriate language.

For any p.e.f of $\mathcal{A}$, we consider the total function (denoted by $\hat{f}$) which is the same as f on its domain, and whose value is constantly zero off $dom(f)$. We prove q.e. for $Th(\mathcal{A})$ in the language L with $+$, $<$ and *these* function symbols, as well as names for the algebraic points. We will call the collection of all algebraic points A_0; notice that the endpoints of $dom(f)$ for any non-total p.e.f are in A_0. The q.e. will follow immediately from the following:

PROPOSITION 6.2: *For any L-term $t(x,\bar{y})$, there are terms $s_1(\bar{y}),\dots s_n(\bar{y})$ and formulas $\rho_i(x,\bar{y})$, $(i=1,2,3)$ so that each ρ_i is a Boolean combination of formulas that say $x=s_j(\bar{y})$, $x<s_k(\bar{y})$, $x>s_\ell(\bar{y})$ and:*

$$\mathcal{A}\models t(x,\bar{y})>0 \Leftrightarrow \rho_1(x,\bar{y}),$$

$$\mathcal{A}\models t(x,\bar{y})<0 \Leftrightarrow \rho_2(x,\bar{y}), \text{ and}$$

$$\mathcal{A}\models t(x,\bar{y})=0 \Leftrightarrow \rho_3(x,\bar{y}).$$

Proof: We define the notion of the *x-depth* of a term (with free variables $x,\bar{y}$ and parameters from A_0) as follows:

1. $dp(x)=dp(s(\bar{y}))=0$ (any $s(\bar{y})$ a term involving parameters from A_0 but no occurrence of x);
2. $dp(-t)=dp(t)$;

3. $dp(t_1 + t_2) = dp(t_1) + dp(t_2)$; and
4. if t is any term involving an occurrence of x and f any p.e., $dp(\hat{f}(t)) = dp(t) + 1$.

Notice that for terms of x-depth 0, the assertion is trivial. We prove the assertion by induction on depth (x will remain fixed throughout). Any term can be written, without changing its depth, as

$$t(x, \bar{y}) = \hat{f}_1(t_1(x, \bar{y})) + \cdots + \hat{f}_k(t_k(x, \bar{y})) + mx + s(\bar{y}).$$

(Here m is an integer.) We may assume none of the f_i's is constantly zero; for each f_i we let $(-a_i, a_i)$ be its domain and $(-b_i, b_i)$ its range. We suppose that we have proved the assertion for terms of smaller depth, and break into 3 cases:

1. $m = 0$. This is the main case. Pick i with b_i maximum. We may assume that $i = 1$ and that $f_1 > 0$. Now let $h(x) = kf_1^{-1}(\frac{1}{k}x)$ for $x \in (-kb_1, kb_1)$, h undefined elsewhere. That is, h is just f_1^{-1} extended to a larger interval. Then we have that h is a p.e., and as long as $|t_1| < a_1$ and $|s(\bar{y})| < kb_1$, that

   $$\mathcal{A} \models t > 0 \Leftrightarrow t_1 + \hat{h}(\hat{f}_2(t_2(x, \bar{y}))) + \cdots + \hat{h}(s(\bar{y})) > 0.$$

   Let $g_i(x) = h(f_i(x))$ for $x \in (-a_i, a_i)$ and $i = 2, \cdots + k$. We have, actually,

   $$\mathcal{A} \models t > 0 \Leftrightarrow \{(|t_1| \geq a_1) \wedge \hat{f}_2(t_2) + \cdots + \hat{f}_k(t_k) + s(\bar{y}) > 0\} \bigvee$$

   $$\{|t_1| < a_1 \wedge [(|s(\bar{y})| < kb_1 \wedge t_1 + \hat{g}_2(t_2) + \ldots \hat{g}_k(t_k) + \hat{h}(s) > 0) \vee s(\bar{y}) \geq kb_1]\}.$$

   The terms involved in the above expression have lower depth than t, so we finish this case by induction. (We deal similarly with the formulas $t < 0$ and $t = 0$.)

2. $m \neq 0$, and at least one $\hat{f}_i(t_i)$ has depth 1. Suppose $dp(\hat{f}_1(t_1)) = 1$, so t_1 is $\ell x + u(\bar{y})$, where $\ell \neq 0$. Let $f_1' = \ell f_1 + m$ on $(-a_1, a_1)$ and

   $$s' = s - \frac{m}{\ell}u.$$

   Then, for

   $$|x + \frac{u}{\ell}| < a_1,$$

we have

$$\hat{f}_1(\ell x + u) + mx + s = \ell \hat{f}_1(x + \frac{u}{\ell}) + m(x + \frac{u}{\ell}) + (s - \frac{m}{\ell}u)$$

which is equal to $\hat{f}'_1(x + \frac{u}{\ell}) + s'$. We have

$$\mathcal{A} \models \{-a_1 < x + \frac{u}{\ell} < a_1 \wedge t = \hat{f}'_1(x + \frac{u}{\ell}) + \hat{f}_2(t_2) \cdots \hat{f}_k(t_k) + s'\} \bigvee$$

$$\{|x + \frac{u}{\ell}| \geq a_1 \wedge (t > 0 \Leftrightarrow \hat{f}_2(t_2) + \cdots + s + mx > 0)\}.$$

With this substitution we reduce to the previous case.

3. Not case 1 or 2. Then t is $\hat{f}_1(t_1) + \cdots + \hat{f}_k(t_k) + mx + s$, where every t_i has positive depth and $m \neq 0$. We deal with this exactly as in the first case, except that now our main replacement term for t is $t_1 + \hat{g}_2(t_2) + \cdots + \hat{g}_k(t_k) + \hat{h}(mx + s)$. This has the same depth as t, but it now falls under either case 1 or case 2, depending on the depth 0 part of t_1. This completes the proof.

∎

COROLLARY 6.3: *In the language described above, $Th(\mathcal{A})$ eliminates quantifiers.*

Proof: Just like for ordered vector spaces. ∎

We should note exactly which axioms of $Th(\mathcal{A})$ were used in the above proof. We need that $\langle A; +, < \rangle$ is an ordered Q-space. For each p.e.f, say with domain $(-a_f, a_f)$, we require an axiom stating that $\hat{f}(x+y) = \hat{f}(x) + \hat{f}(y)$ if $x, y, x+y \in (-a_f, a_f)$ and $\hat{f}(x) = 0$ for $|x| \geq a_f$. We need the various relations that hold among the p.e.'s; for example that $\hat{f}(x) + \hat{g}(x) = \widehat{f+g}(x)$ for $x \in dom(f) \cap dom(g)$, or if we have that $dom(f) \subseteq dom(g)$ and that $f(x) = g(x)$ on $dom(f)$ we need to include $\hat{f}(x) = \hat{g}(x)$ for $-a_f < x < a_f$ as an axiom. Finally, the equations and inequalities among quantifier-free terms should be included. The collection of all such sentences therefore provides an axiomatization of $Th(\mathcal{A})$.

We define $dom(t)$ (the "real domain" of t) for any L-term $t(\bar{x})$. First, if $t'(x_1, \ldots, x_{k+1}) = t(x_1, \ldots, x_k)$, then $dom(t') = dom(t) \times A$. For x a variable, $dom(x) = A$; for $a \in A_0$, regarded as a 0-ary term, the notion is vacuous. If $t(\bar{x}) = t_1(\bar{x}) + t_2(\bar{x})$, then $dom(t) = dom(t_1) \cap dom(t_2)$. Finally, if $t(\bar{x}) = \hat{f}(s(\bar{x}))$ for f a p.e., $dom(t) = \{\bar{a} \in dom(s) \colon s(\bar{a}) \in dom(f)\}$. Notice that

$dom(t(x_1, \ldots, x_k))$ is always 0-definable in our chosen language, and is in fact an open neighborhood of $\bar{0}$ in A^k.

We recall the division ring D associated naturally with any A satisfying the hypotheses of Theorem 6.1. We identify two p.e.'s f and g if they agree on some (any) nonzero element of $dom(f) \cap dom(g)$; the equivalence classes are the elements of D. We will write d_f for the element of D corresponding to the p.e. f. Choose an $|A|^+$-saturated elementary extension $\mathcal{A}'$ of $\mathcal{A}$. Let V be the set of all infinitesimal elements of $\mathcal{A}'$ together with 0. It is readily seen that V inherits an ordered D-space structure, call it $\mathcal{V}$, from $\mathcal{A}'$, with $d_f = f|V$. Given any L-term $t(\bar{x})$, we let $t^*(\bar{x})$ be the corresponding term of the vector space language, i.e., replace any $\hat{f}$ by d_f. Similarly, for any formula $\chi(\bar{x})$ of the form

$$\bigwedge t_i(\bar{x}) > 0 \wedge \bigwedge s_j(\bar{x}) = 0$$

of the language of $\mathcal{A}$ again involving no constant symbols except 0, we write χ^* for the corresponding formula of the ordered vector space.

We aim to find an embedding σ of A into V so that for every L-term t and $\bar{a} \in dom(t)$ we have

$$\mathcal{A} \models t(\bar{a}) > 0 \Rightarrow \mathcal{V} \models t^*(\sigma(\bar{a})) > 0.$$

Also we show the corresponding result if $\mathcal{A} \models s(\bar{a}) = 0$ for a term s. This amounts to realizing a type p inside $\mathcal{V}$ of a collection of formulas with $|A|$ variables, where each formula has the form $\chi^*(\bar{x})$ for χ as described above. So we will prove by induction on the number of variables $\bar{x}$ that if $\mathcal{A} \models \exists\bar{x}[\chi(\bar{x}) \wedge \bigwedge \bar{x} \in dom(t_i) \cap dom(s_j)]$, then $\mathcal{V} \models \exists\bar{x}\chi^*(\bar{x})$. We let χ' abbreviate $\chi(\bar{x}) \wedge \bigwedge \bar{x} \in dom(t_i) \cap dom(s_j)$. There is nothing to do if there are no free variables in χ. The consistency of the relevant type is immediate from the following simple proposition:

PROPOSITION 6.4: *For $\chi(\bar{x})$ as above, if $\mathcal{A} \models \exists\bar{x}\chi'(\bar{x})$, then for any $\epsilon > 0$, $\mathcal{A} \models \exists\bar{x}(\chi'(\bar{x}) \wedge \bigwedge |x_i| < \epsilon)$.*

Proof: If this is false, choose i minimal so that

$$\mathcal{A} \models (\chi'(\bar{x}) \wedge \bigwedge_{j<i} |x_j| < \epsilon) \Rightarrow |x_i| \geq \epsilon.$$

Consider the set

$$\{x_i\colon \exists x_1 \cdots x_{i-1}x_{i+1} \cdots x_n(\chi'(\bar{x}) \wedge \bigwedge_{j<i} |x_j| < \epsilon)\}.$$

This is nonempty but bounded away from zero. We may assume it has a positive element and then let e be the infimum of the positive elements of this set. Pick any a_i with $0 < a_i < 2e$ in this set. We can find $\bar{a}$ with ith entry a_i so that $\mathcal{A} \models \chi'(\bar{a}) \wedge \bigwedge_{j<i} |a_j| < \epsilon$. Now notice that for any term $t(\bar{x})$ with no constants, if $\bar{a} \in dom(t)$, so is $\frac{1}{2}\bar{a}$, and $t(\frac{1}{2}\bar{a}) = \frac{1}{2}t(\bar{a})$. This implies that $A \models \chi'(\frac{1}{2}\bar{a})$. But clearly $A \models \bigwedge_{j<i} |\frac{1}{2}a_j| < \epsilon$ too, so $\frac{1}{2}a_i$ is in the set; this contradiction establishes the proposition. ∎

This proposition makes the consistency of all the $\chi^*(\bar{x})$'s in p clear, and this leads instantly to the existence of the function σ required. It is also immediate that, given $\bar{a}$ from A, if $A \models \exists\bar{x}\chi'(\bar{x}, \bar{a})$, then $\mathcal{V} \models \exists\bar{x}\chi^*(\bar{x}, \sigma(\bar{a}))$. We identify A with $\sigma(A)$ in what follows.

It is clear what reduct $\mathcal{V}'$ of the vector space $\mathcal{V}$ we will take to verify Theorem 6.1. We keep $+$ and $<$, and name all the points in A_0. Finally, for any p.e. f of A with $dom(f) = (-a, a)$, the function f^V so that $f^V(x) = d_f(x)$ for $-a < x < a$ and 0 elsewhere is 0-definable in the ordered vector space with a named, so this is the interpretation of $\hat{f}$ in $\mathcal{V}'$. It is obvious that $\mathcal{V}'$ satisfies the axioms listed above for $Th(\mathcal{A})$, and now from the quantifier elimination for $\mathcal{A}$, we see that $\mathcal{A} \preceq \mathcal{V}'$. This completes the proof of Theorem 6.1. ∎

Now, as promised, we can easily characterize group-intervals whose theory is o-minimal and linear (modular). The result is:

THEOREM 6.5: *Let T be the theory of an o-minimal CF group-interval. Then there is an interval of an ordered vector space $\mathcal{I}$ such that T is a reduct of $Th(\mathcal{I})$. If T is in fact modular, T is interdefinable with $Th(\mathcal{I})$.*

Proof: We know how to construct a full group out of any $\mathcal{M} \models T$. Then we know how to embed said group into an ordered vector space. The division ring for this space remains the one we have for $\mathcal{M}$. It is clear that anything definable in $\mathcal{M}$ is definable in the vector space by Proposition 4.2 above. In the modular case, we have (Lemma 4.3) that in $\mathcal{M}$, for every p.e. f, either f or f^{-1} is total. From this and the quantifier elimination for $\mathcal{V}|I$ (as pointed out on page 9), it is clear that $\mathcal{M}$ gains no new structure when it is identified with $I \subseteq V$. ∎

Example 6.6: Let $\mathcal{R}'$ be a nonstandard elementary extension of the real field, $\alpha \in R'$ be infinite, and f the p.e. defined on $(-\alpha^{-1}, \alpha^{-1})$ by $f(x) = \alpha x$. Then $Th\langle R'; +, <, f\rangle$ is linear in our sense, and a reduct of an ordered vector space over $D = Q(\alpha)$. However, if we let $\mathcal{A} \preceq \langle R'; +, <, f\rangle$ omit the type $\{x > 1, x > 2, x > 3, \ldots\}$, we cannot expand $\mathcal{A}$ itself to an ordered vector space over D. The group that arises in our construction of the last section from $\langle R'; +, <, f\rangle | [-1, 1]$ will be such an $\mathcal{A}$.

References

[vdD] L. van den Dries, *Tame topology and σ-minimal structures*, preliminary version, 1991.

[HL] E. Hrushovski and J. Loveys, *Weakly minimal groups of bounded exponent*, in preparation.

[KPS] J. Knight, A. Pillay and C. Steinhorn, *Definable sets in ordered structures II*, Trans. Amer. Math. Soc. **295** (1986), 593–605.

[P1] Y. Peterzil, *Some definability questions in structures over the reals and in general o-minimal structures*, Ph.D. thesis, University of California, Berkeley, 1991.

[P2] Y. Peterzil, *Zilber's Conjecture for some o-minimal structures over the reals*, to appear in Ann. Pure Appl. Logic.

[P3] Y. Peterzil, *Constructing a group-interval in o-minimal structures*, preprint.

[Pi] A. Pillay, *On groups and fields definable in o-minimal structures*, J. Pure Appl. Algebra **53** (1988), 239–255.

[PSS] A. Pillay, P. Scowcroft and C. Steinhorn, *Between groups and rings*, Rocky Mountain J. **9** (1989), 871–885.

[PS] A. Pillay and C. Steinhorn, *Definable sets in ordered structures I*, Trans. Amer. Math. Soc. **295** (1986), 565–592.